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Unsteady flow in straight alluvial streams: modification of individual dunes

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Changes in dune properties due to a sudden change in the water discharge are analysed. The transport of sediment is assumed to occur mainly as bed load and the square of the Froude number must be much smaller than unity. The theory is based on similarity in the bed-shear distribution close to the dune top just before and just after the change in the water discharge. The model is found to agree with laboratory experiments. Further, by use of this model, flow in a river where the water discharge oscillates weakly around a constant mean value has been treated. Hereby the variation in the phase differences between sediment transport, water depth and water discharge with the period of the oscillation is calculated. The results agree qualitatively with observations.

1. Introduction

Unsteady flow in streams with movable beds is generally much more complex than unsteady flow over a fixed bed. One of the main reasons for this is that dunes develop in the lower flow regime if the bed is erodible. Behind the crests of the dunes, an expansion loss occurs which changes the flow resistance and rate of sediment transport radically compared with the flow in a river with a fixed bed. In a steady flow, dunes develop whose dimensions vary with the hydraulic conditions (specific water discharge and slope of the river) and the sediment properties (mean diameter, fall velocity and standard deviation). If the hydraulic parameters change with time so do the geometrical dimensions of the dunes (height and length). However, if the water discharge changes rapidly enough, the geometrical dimensions of the bed forms are not related to the instantaneous hydraulic conditions, because it takes time for the dunes to change their geometrical properties. This time lag is important because the expansion loss behind the crests of the dunes strongly depends on the dune geometry. Therefore, to describe the variation in water level, flow velocity and sediment transport with time, accurate knowledge of the unsteady behaviour of dunes is needed.

Earlier work on the same subject has been carried out by Allen (1976*a*, *b*), who splits the changes in the configurations into two different parts: (i) changes in the properties of the individual dunes; (ii) replacement of the original dunes by others better adjusted to the new hydraulic conditions. In his work, Allen normally assumes that the first process takes place much faster than the second. A discussion of the two processes is carried out in the 1976*b* paper. However, Allen introduces several empirical constants and assumptions. First, the rate of change due to modification of the individual dunes is assumed to be proportional to both the instantaneous migration

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velocity of the dune and the total change in the dune dimensions; a constant of proportionality is also introduced. Second, he assumes that the rate of change in the exchange process is proportional to the total change in the dune dimensions, and also inversely proportional to a time scale T called the life span, which is defined as the interval between creation and destruction. During this time, the bed form has travelled a distance CL in the downstream direction, where C is called the excursion and L is the wavelength of the single bed form. The excursion and life span are related by

$$\frac{1}{L}\int_0^T a(t)\,dt = C, \tag{1}$$

where $a(t)$ is the instantaneous migration velocity of the single bed form. In Allen's analysis C must be determined empirically.

In the present paper, the stochastic nature of the bed forms is neglected, so the present paper must be regarded as an attempt to give a physical understanding of the process which modifies the dimensions of the individual dunes as the hydraulic conditions change. Hence the bed is assumed to be covered by regular dunes which all have the same wavelength and height. For simplicity, the dunes are taken as two-dimensional and their shape is approximated by the well-known triangular form with a slightly curved upstream surface and a downstream slope approximately equal to the angle of repose.

2. Changes in the dune height due to a sudden change in the water discharge

In this section a model is developed which is able to describe the initial variation in the dune height with time after a sudden change in the water discharge from one value Q_1 to another value Q_2. The transport of sediment is assumed to occur mainly as bed load, i.e. the transport of sediment responds to spatial changes in the tractive stress without any lag, assuming the inertia of the bed particles to be negligible, which will be the case except for very coarse grains, see Fredsøe (1976).

Before the water discharge is changed, the dune pattern is supposed to be in equilibrium, which means that the dunes are travelling in the downstream $(+x)$ direction with velocity a_1 without changing form:

$$h = h(x - a_1 t), \tag{2}$$

where h is the bed elevation and t the time. The continuity equation of sediment is

$$\partial q_1/\partial x = -(1-n)\,\partial h/\partial t, \tag{3}$$

where n is the porosity and q_1 the sediment discharge (Engelund & Hansen 1972). Combining (1) and (2) yields

$$\frac{\partial q_1}{\partial x} = a_1(1-n)\frac{\partial h}{\partial x} \quad \text{or} \quad q_1 = a_1(1-n)\,h + \text{const.}, \tag{4}$$

where the constant equals zero if the bed elevation is put equal to zero in troughs, in which the bed load always vanishes; see figure 1.

After the change in the water discharge, the sediment discharge will be changed to q_2 and the migration velocity of the fronts of the dunes to a_2. The downstream slope

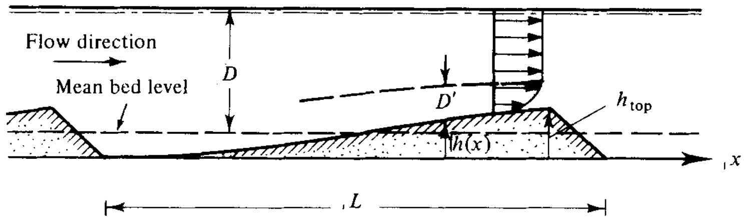

FIGURE 1. Definition sketch.

of a dune just after the crest is approximately equal to the angle of repose of the bed material. Hence, for geometrical reasons, the new migration velocity a_2 is determined by

$$a_2 = [q_2/(1-n)\,h]_{top}, \tag{5}$$

where the quantities on the right-hand side are evaluated at the top of the dune.

The initial change in the dune height is calculated as the initial change in height at the top. Normally, the dune height is defined as the difference between the height at the top and the height in the trough. However, in the trough the shear stress is very small, owing to the separation behind the crest; this means that the rate of bed load here is close to or equal to zero, so the level of the bed is not changing.

Now the initial variation in the dune height is found from

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + a_2 \frac{\partial h}{\partial x}$$

or, by use of (3) and (5),

$$\frac{dh}{dt} = -\frac{1}{(1-n)}\frac{\partial q_2}{\partial x} + \frac{q_2}{h(1-n)}\frac{\partial h}{\partial x}, \tag{6}$$

where all the quantities are evaluated at the top of the dune.

As the sediment is assumed to be mainly bed load, the dimensionless rate of sediment load can be related to the local value of the Shields parameter θ by

$$\Phi = \Phi(\theta), \quad \text{where} \quad \Phi = \frac{q}{[(s-1)\,gd^3]^{\frac{1}{2}}}, \quad \theta = \frac{\tau}{(s-1)\,gd}, \tag{7}$$

and where s is the relative density of the sediment, d the mean grain diameter, g the acceleration due to gravity and τ the local value of the part of the shear stress which acts directly as a skin friction on the surface of the dune. Introducing (7) into (6), this can be written as

$$\frac{1-n}{[(s-1)\,gd^3]^{\frac{1}{2}}}\frac{dh}{dt} = -\frac{d\Phi_2}{d\theta_2}\frac{d\theta_2}{dx} + \frac{\Phi_2}{h}\frac{\partial h}{\partial x}, \tag{8}$$

where all the quantities are still evaluated at the dune top; Φ_2 is the dimensionless rate of bed load and θ_2 the Shields parameter just after the water discharge has been changed to Q_2.

Immediately after the water discharge has been changed, the geometrical properties of the bed remain unchanged. As the water discharge is changed, the current velocity is also changed, but as the geometrical boundary remains unchanged for a Froude number far from unity, it is assumed that the bed shear stress distribution is similar to the distribution in the equilibrium situation before the discharge is changed. This implies that an increase in the Shields parameter θ at the top of the dune by a factor Δ, defined by

$$\theta_2 = \Delta\theta_1, \tag{9}$$

gives an increase in the longitudinal variation of the bed shear stress by the same factor, so

$$\partial\theta_2/\partial x = \Delta\,\partial\theta_1/\partial x. \tag{10}$$

The longitudinal variation in the bed shear stress in the equilibrium situation is related to the dune geometry, and by using (4) and (7),

$$\frac{\partial\theta_1}{\partial x} = \frac{\partial\Phi_1}{\partial x}\Big/\frac{d\Phi_1}{d\theta_1} = \frac{a_1(1-n)}{[(s-1)\,gd^3]^{\frac{1}{2}}}\,\frac{\partial h}{\partial x}\Big/\frac{d\Phi_1}{d\theta_1}. \tag{11}$$

The migration velocity a_1 is determined by (4), thus (11) can be written as

$$\frac{\partial\theta_1}{\partial x} = \Phi_1\frac{1}{h}\frac{\partial h}{\partial x}\Big/\frac{d\Phi_1}{d\theta} \simeq \Phi_1\Big/\left(L\,\frac{d\Phi_1}{d\theta}\right), \tag{12}$$

in which the expression $h^{-1}\partial h/\partial x$, because of the nearly triangular form of the dune, has been approximated by $1/L$, where L is the dune length. Inserting (10) and (12) into (8), the initial change in dune height is found to be

$$\frac{dh}{dt} = \frac{[(s-1)\,gd^3]^{\frac{1}{2}}}{(1-n)\,L}\left\{1-\Delta\,\frac{(\Phi^{-1}d\Phi/d\theta)_2}{(\Phi^{-1}d\Phi/d\theta)_1}\right\}\Phi_2, \tag{13}$$

where all quantities still are evaluated at the top of the dune.

In the present paper, the following sediment transport formula is used:

$$\Phi(\theta) = 5\left\{1+\left(\frac{0\cdot 267}{\theta-\theta_c}\right)^4\right\}^{-\frac{1}{4}}(\theta^{\frac{1}{2}}-0\cdot 7\theta_c^{\frac{1}{2}}) \tag{14}$$

(Engelund & Fredsøe 1976). Bed load transport formulas like those of Einstein (1950) or Meyer-Peter & Müller (1948) do not deviate much from (14), and can be used in the analysis with approximately the same quantitative results.

Equation (14) was originally derived for the sediment transport on a plane bed. However, as long as the effect of inertia and gravity is weak, it can also be applied locally on the dune surface. As mentioned above, the effect of inertia is weak. Further, on the *upstream* part of the dunes, the effect of gravity is very small; as described by Fredsøe (1974), the local value of the Shields parameter should be corrected by $0\cdot 1S$, where S is the local slope of the bed, which is of order h/L. For dunes, this ratio is about 0·01–0·07, so the correction to the Shields parameter is of order 0·001–0·01 and can therefore normally be disregarded.

3. Comparison with experiments

Experiments concerning the subject have been carried out by Gee (1973). In his experiments, the water discharge was abruptly changed from one value to another, after which change the gradient of the flow and the water discharge were kept constant. Gee then measured the variations in the water level due to the changes in the friction factor caused by the dunes changing form. In order to compare these experiments with the present theory, a relation between the changes in the dune properties and the depth is needed. This relation is obtained if the friction factor f for flow over a dune-covered bed is known because the depth and the friction factor are related by

$$f = \{2Q^{-2}gI\}\, D^3, \tag{15}$$

where D is the water depth, I the gradient of the flow and Q the specific discharge. Equation (15) is a form of the Darcy–Weisbach equation. In flow over a dune-covered bed, the friction factor is normally split into two parts: f', which is the contribution to the friction factor from the pure skin friction, and f'', which is the similar contribution from the expansion loss behind the crest of the dune. This last contribution has recently been treated as a Carnot loss by Engelund (1978), who obtained the relation

$$f = f' + f'' = f' + 2{\cdot}5 \exp(-2{\cdot}5h/D)\, h^2/DL; \tag{16}$$

f' is found from the relations

$$(2/f')^{\frac{1}{2}} = 6 + 2{\cdot}5 \ln (D'/k) \quad \text{and} \quad f'/f = D'/D \tag{17}$$

given by Einstein (1950). k is the sand roughness. f' can be taken as a constant in the unsteady case too, because even great changes in the water depth only slightly affect the numerical value of f'.

Like the friction factor f, the dimensionless shear stress θ is split up into θ' and θ''. Here, θ' can be calculated from (17) if the mean flow velocity V is known. The idea behind (17) is that a boundary layer of thickness D' is formed along the surface of the dune, while the velocity distribution outside this layer is very nearly uniform (like flow between two converging walls); see figure 1. The value of θ' obtained from (17) can be interpreted as a sort of mean value along the dune surface because the pure skin friction varies along the dune. Close to the top of the dune, the dimensionless bed shear stress must approach a value close to θ' as calculated from (17), but based on the local value of the mean flow velocity.

Gee carried out experiments with two grain sizes: $d = 0{\cdot}30$ mm and $d = 1{\cdot}00$ mm. The specific gravity of the sediment was 2·65. However, in all the experiments using fine-grained sediments, the Froude number crosses the critical value 1 during the transition. When the square of the Froude number is close to one, the undulations on the water surface will be very large, especially if the bed is covered by irregularities as in the present case. In this situation the assumption of similarity in the bed shear stress before and after the change in water discharge will break down. Further, because of the large flow velocities combined with the small mean diameter of the sand, the dimensionless bed shear stress θ' is rather large. At the greatest value of the water discharge in a run, θ' is typically about 0·40. Hence the ratio between transport of suspended sediment and bed load is found to be close to one when the method of Engelund & Fredsøe (1976) is used to split the total load into suspended load and bed load. The suspended load responds to spatial changes in the tractive stress with a certain lag because a grain in suspension takes some time to settle after it has been picked up from the bed. In this case the theory developed in § 2 is not valid.

In tables 1 and 2 the relevant data concerning the coarse sand experiments are given together with the calculated value of Δ. The bed shear stress is calculated from (17).

Figure 2 shows a comparison between the experiments and theory. D_0 is depth just after the water discharge has been changed (table 2), while D is the depth at a

Run	h (cm)	L (m)	θ'	D (cm)
1	0·49	2·09	0·077	6·2
2	1·77	1·33	0·141	14·6
3	0·98	2·09	0·116	10·0
5	0·55	2·43	0·085	8·2
6	1·95	1·14	0·105	14·6

TABLE 1. Data for the equilibrium case (obtained from Gee 1973).

Run	D_0 (cm)	θ'	Δ	C
$1 \to 2$	12·5	0·174	2·26	4·0
$2 \to 3$	11·4	0·094	0·67	2·5
$5 \to 6$	12·5	0·137	1·61	5·8

TABLE 2. Transition data just after the water discharge has been changed (obtained from Gee 1973).

time t after the change in water discharge. The pairs of full lines yield the theoretical results; the initial slope is found by using (13), (15) and (16). It is assumed that the initial change in the ratio between dune height and dune length is given by

$$\frac{d(h/L)}{dt} = \frac{1}{L}\frac{dh}{dt}, \tag{18}$$

which seems reasonable as changes in the dune height can occur immediately, while it takes some time for a dune to change its length since this involves the formation and disappearance of individual dunes. However, the choice given by (18) is not decisive, as the form drag mainly depends on the height of the dune as seen from (16). Furthermore the relative change in the dune height will normally be much larger than the relative change in the dune length, as in the data in the tables.

The horizontal full lines in figure 2 yield the final state, where the dune height and length are again in equilibrium. This state is calculated from (15) and (16) using the equilibrium data for the dune height and length.

The dashed lines in figure 2 are the curves which are obtained by matching the initial variation in the ratio h/L with the values in the equilibrium situation by means of an exponential curve of the form

$$\frac{h}{L} = \frac{h_1}{L_1} + \left(\frac{h_0}{L_0} - \frac{h_1}{L_1}\right) \exp\left(\frac{-L_0^{-1}\,dh/dt}{h_1/L_1 - h_0/L_0}\,t\right), \tag{19}$$

where the subscript 0 denotes the equilibrium state before the water discharge and gradient has been changed, and the subscript 1 the equilibrium state after the change.

The agreement between theory and experiment is satisfactory especially for the first two runs. However, in run $5 \to 6$ the agreement in the initial slope is also good. It seems in this experiment to be a little difficult to determine the depth D_0 as the scatter at $t = 0$ is rather large.

An excursion C analogous to that given by Allen [see (1)] can be defined by putting T equal to the time it takes for the dune to change its height by $\frac{1}{2}\Delta h$, where Δh is the

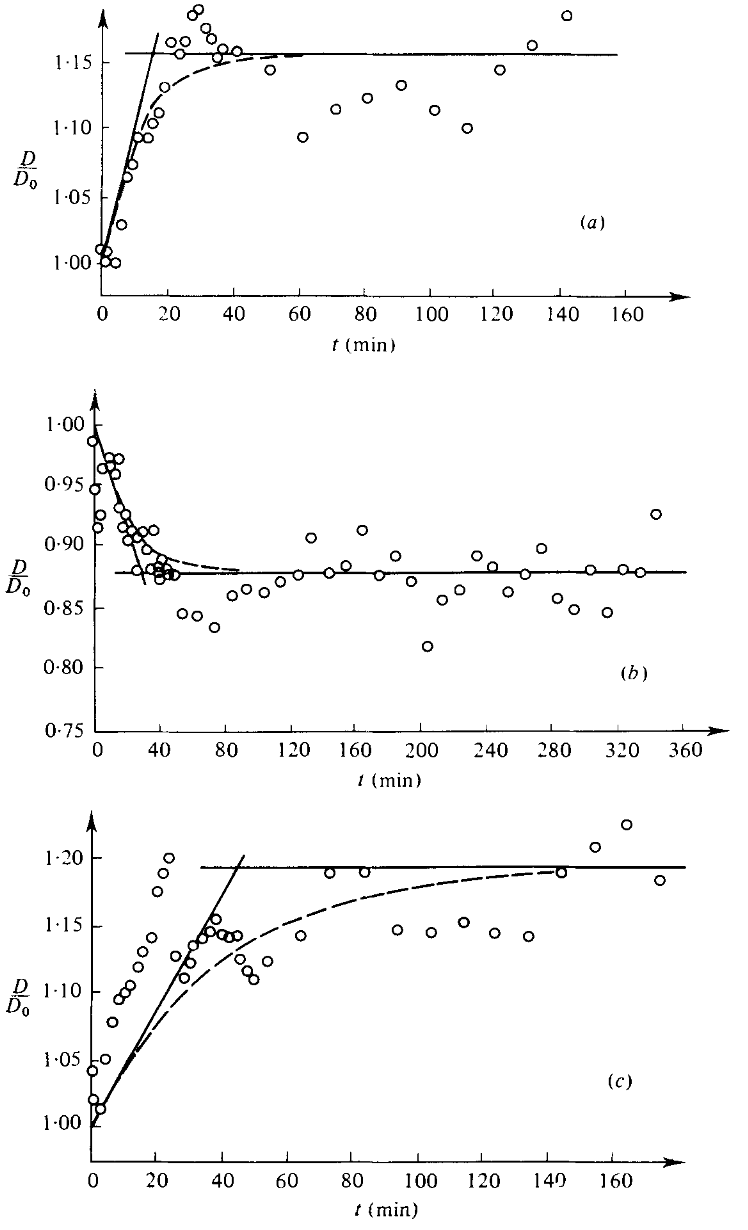

FIGURE 2. Changes in water depth with time after a sudden change in water discharge. The circles are the data obtained by Gee (1973). Using the numbering of Gee, (*a*) describes run $1 \rightarrow 2$, (*b*) $2 \rightarrow 3$ and (*c*) $5 \rightarrow 6$.

difference between the new and the former equilibrium wave heights. The calculated values of C are incorporated in table 2, and are seen to be somewhat larger than unity.

4. Linearization of the equations

Initial variations in bed shear stress, dune height and water depth

By use of the formulae developed in § 2, it is possible to calculate the behaviour of a river bed as the water discharge varies with time. If the changes in the discharge are relatively large a numerical method must be used.

In this section, the simple case where small variations are superposed on a constant water discharge is calculated analytically. Longitudinal variations in specific water discharge and in water depth are neglected in the following discussion. Thus such factors as sedimentation or erosion of the river bed and the effect of a propagating flood wave are disregarded. In principle the flood-wave effect can be incorporated in the analysis, but will increase the calculations considerably. However, as will be seen later, the time scale for changes in the dune dimensions is normally much larger than a typical time scale for a flood wave, so the two problems may be treated independently.

First we linearize the formulae developed in § 2; the water discharge is assumed to change from $\bar{Q}$ to $\bar{Q}+dQ$ instantaneously. This involves a small change $d\theta'$ in θ, so Δ, defined in (10), can be written as

$$\Delta = 1+d\theta'/\theta'. \tag{20}$$

As in § 3, the dimensionless bed shear stress at the top is approximated by the effective Shields parameter θ'.

Introducing (20) into (13) yields at $t = 0$

$$\frac{dh}{dt} = -\frac{[(s-1)\,gd^3]^{\frac{1}{2}}}{(1-n)\,L}\,\Phi\left\{\frac{1}{\theta'}+\frac{1}{f}\frac{df}{d\theta'}\right\}d\theta' = \alpha_1\,d\theta', \tag{21}$$

where

$$f = \Phi^{-1}\,d\Phi/d\theta'; \tag{22}$$

$d\theta'$ at $t = 0$ is related to the change in water discharge dQ. The relationship can be found from

$$\frac{\theta'}{\theta} = \frac{f'}{f} = \frac{f-f''}{f}, \tag{23}$$

which gives

$$\frac{d\theta'}{\theta'} = \frac{d\theta}{\theta}-\frac{\theta}{\theta'}\,d\left(\frac{f''}{f}\right) = \frac{dD}{D}-\frac{\theta}{\theta'}\,d\left(\frac{f''}{f}\right); \tag{24}$$

f''/f is found from (15) and (16) to be

$$\frac{f''}{f} = \frac{2\cdot5}{2gI}\exp\left(-2\cdot5\,\frac{h}{D}\right)\frac{h^2}{DL}\frac{Q^2}{D^3}. \tag{25}$$

Therefore, at small values of t,

$$d\left(\frac{f''}{f}\right) = \frac{\partial(f''/f)}{\partial Q}\,dQ+\frac{\partial(f''/f)}{\partial D}\,dD+\frac{\partial(f''/f)}{\partial h}\,dh, \tag{26}$$

where the coefficients are easily obtained from (25).

The changes in water depth D with time are calculated as follows. As in (18), the variations in wavelength are disregarded at small values of t, which yields

$$dD = \frac{\partial D}{\partial Q}\,dQ + \frac{\partial D}{\partial h}\,dh, \tag{27}$$

where $\partial D/\partial Q$ is calculated from (15) and (16) to be

$$\frac{\partial D}{\partial Q} = \frac{4gID^4}{Q^3}\Big/\left\{\frac{8}{Q^2}\,gID^3 - f' - 2{\cdot}5\left(\frac{h}{L}\right)^3\left(\frac{L}{D}\right)^2\exp\left(-2{\cdot}5\,\frac{h}{D}\right)\right\}. \tag{28}$$

The change in water depth with dune height is also calculated from (15) and (16) to be

$$\frac{\partial D}{\partial h} = \frac{2{\cdot}5(h/L)^2\exp\left(-2{\cdot}5h/D\right)\left[2L/h - 2{\cdot}5L/D\right]}{(8/Q^2)\,gID^3 - f' + 6{\cdot}25(h/D)^2\,(h/L)\exp\left(-2{\cdot}5h/D\right)}. \tag{29}$$

The variation in the dune height at small times is found from (21), thus (27) yields

$$dD = \frac{\partial D}{\partial Q}\,dQ + \alpha_1\frac{\partial D}{\partial h}\,t\,d\theta'. \tag{30}$$

Inserting (21) and (30) into (26), we obtain

$$d(f''/f) = \gamma_1\,dQ + \gamma_2\,d\theta' t, \tag{31}$$

in which

$$\gamma_1 = \frac{\partial(f''/f)}{\partial Q} + \frac{\partial(f''/f)}{\partial D}\,\frac{\partial D}{\partial Q} \tag{32}$$

and

$$\gamma_2 = \frac{\partial(f''/f)}{\partial D}\,\alpha_1\frac{\partial D}{\partial h} + \frac{\partial(f''/f)}{\partial h}\,\alpha_1. \tag{33}$$

At small values of t, (24) now reads

$$d\theta' = \rho_1\,dQ + \rho_2\,dQt, \tag{34}$$

where

$$\left.\begin{aligned}\rho_1 &= \frac{\theta'}{D}\frac{\partial D}{\partial Q} - \theta\gamma_1\\ \rho_2 &= \rho_1\left\{\theta'\,\frac{\partial D}{\partial h}\frac{\alpha_1}{D} - \gamma_2\,\theta\right\}.\end{aligned}\right\} \tag{35}$$

and

Introducing (34) and (18) into (21), the initial variation in h/L can be written as

$$\frac{d}{dt}\left(\frac{h}{L}\right) = \alpha_1\rho_1\,dQ = \alpha\,dQ. \tag{36}$$

Finally, the initial variation in water depth is found from (27) to be

$$dD = \left\{\frac{\partial D}{\partial Q} + \frac{\partial D}{\partial h}\,\alpha Lt\right\}dQ. \tag{37}$$

The equilibrium states for dune height, water depth and shear stress

In order to obtain a general expression like (19), it is necessary to know the value of h/L in the equilibrium situation as function of the hydraulic parameters and sediment properties.

For a river with constant specific water discharge, the dimensions of the bed forms are assumed to be uniquely determined by the slope and the sediment properties (mean grain diameter, standard deviation and fall velocity). If the dune dimensions are uniquely determined, so are the flow resistance and total load. This assumption just states that identical rivers (same slope, sediment properties and specific water discharge) possess identical bed forms.

For a dune-covered bed, it has been shown by Fredsøe (1975) that the following relation is valid:

$$h/L = 0{\cdot}12(1-0{\cdot}06/\theta-0{\cdot}4\theta)^2. \tag{38}$$

This relation is close to the plots given by Yalin (1972). Hence the change in the equilibrium value of h/L is given by $d(h/L) = \beta_1\, d\theta_e$, where

$$\beta_1 = 0{\cdot}24(1-0{\cdot}06/\theta-0{\cdot}4\theta)(0{\cdot}06/\theta^2-0{\cdot}4). \tag{39}$$

In (39), $d\theta_e$ stands for the change in θ between the two equilibrium situations and is found as follows. In the equilibrium situation, the friction factor f has been found to vary with θ as (Fredsøe 1975)

$$f = \tfrac{1}{18}(1-0{\cdot}06/\theta-0{\cdot}4\theta)^2. \tag{40}$$

From (15) we further have the relation

$$Q^2 \sim \theta^3/f$$

for a fixed value of the gradient. Combining this relation with (29), we obtain

$$Q = \text{const.} \times \frac{\theta^{\frac{3}{2}}}{1-0{\cdot}06/\theta-0{\cdot}4\theta},$$

which yields

$$\frac{dQ}{Q} = \left\{\frac{3}{2} - \frac{0{\cdot}06/\theta-0{\cdot}4\theta}{1-0{\cdot}06/\theta-0{\cdot}4\theta}\right\}\frac{d\theta_e}{\theta} = K_1\frac{d\theta_e}{\theta}. \tag{41}$$

Now (39) can be written as

$$d\left(\frac{h}{L}\right) = \frac{\beta_1\theta}{K_1 Q}\, dQ = \beta dQ. \tag{42}$$

Using (36) and (42) the linearized version of (19) is

$$d\left(\frac{h}{L}\right) = \beta\left(1-\exp\left[-\frac{\alpha}{\beta}t\right]\right) dQ. \tag{43}$$

The water depth in the equilibrium situation is found from the relation

$$\theta = DI/(s-1)\,d,$$

which combined with (41) yields

$$dD_e = \frac{D}{QK_1}\, dQ. \tag{44}$$

Hence the variation in water depth with time is approximated by

$$dD = \frac{\partial D}{\partial Q}\, dQ + d_1\left[1-\exp\left(-d_1^{-1}\alpha L\frac{\partial D}{\partial h}t\right)\right] dQ, \tag{45}$$

where

$$d_1 = D/QK_1 - \partial D/\partial Q.$$

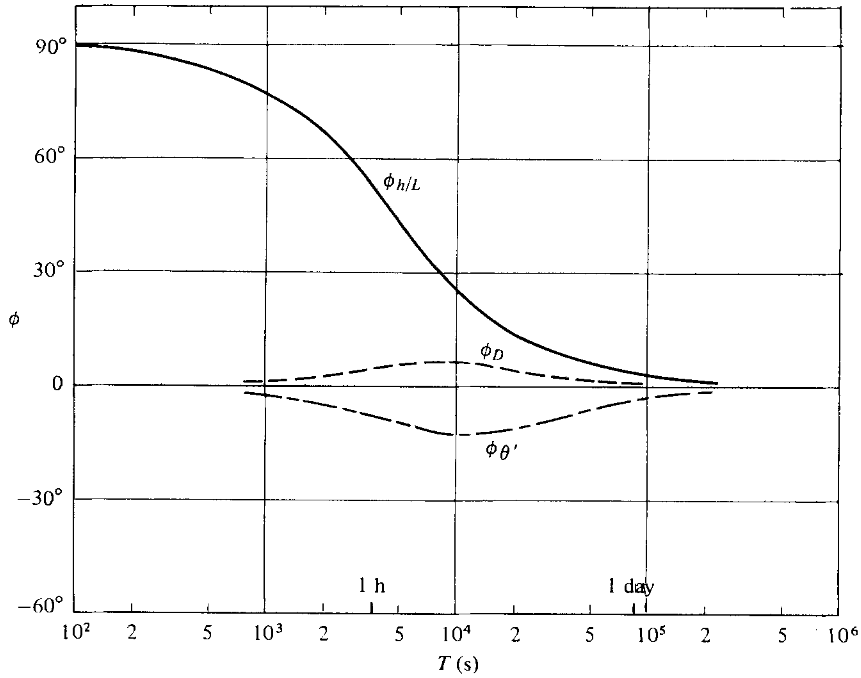

FIGURE 3. Variation in phase with the period T for $Q = 0{\cdot}1\,\mathrm{m^2/s}$, $I = 3 \times 10^{-3}$ and $d = 1{\cdot}00\,\mathrm{mm}$, which yields $h = 3{\cdot}2\,\mathrm{cm}$, $L = 0{\cdot}60\,\mathrm{m}$ and $D = 0{\cdot}15\,\mathrm{m}$.

Normally the rate of sediment transport is related to the value of the effective bed shear stress θ'; see for instance Engelund & Hansen (1972). In order to investigate the changes in sediment transport with time, the variation in θ' is finally considered.

For a dune-covered bed in equilibrium, Engelund & Hansen (1972) obtained the relation

$$\theta' = 0{\cdot}06 + 0{\cdot}4\theta^2. \tag{46}$$

Hence, in the equilibrium case

$$d\theta'_e = 0{\cdot}8\theta\, d\theta_e = 0{\cdot}8\theta^2 \frac{dD_e}{D},$$

or, by use of (41),

$$d\theta'_e = \frac{0{\cdot}8\theta^2}{QK_1} dQ = \lambda dQ.$$

The variation in θ' with time is then given by

$$d\theta' = \rho_1 dQ + (\lambda - \rho_1)\left[1 - \exp\left(\frac{-\rho_2}{\lambda - \rho_1} t\right)\right] dQ. \tag{47}$$

5. Weakly periodically varying water discharge

From (43), (45) and (47) it is possible to calculate the variations in h/L, the water depth and the sediment transport if the water discharge varies weakly with time. In the following we consider the case

$$dQ(t) = dQ_0 \sin(\omega t), \tag{48}$$

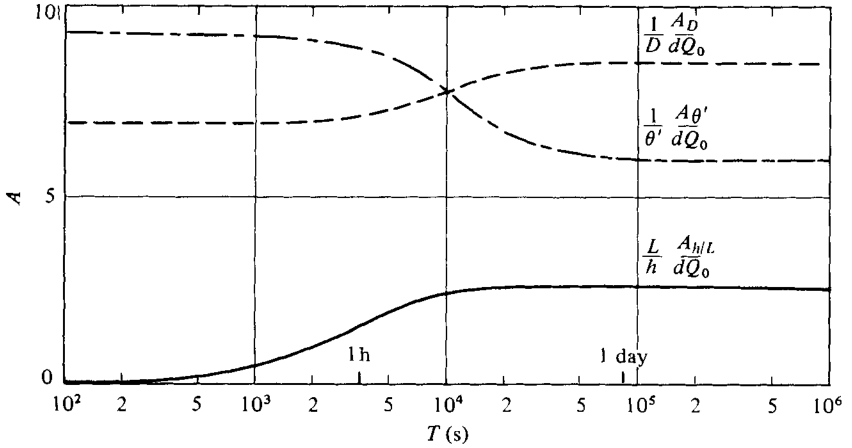

FIGURE 4. Variation in relative amplitudes with T. The data are the same as in figure 3.

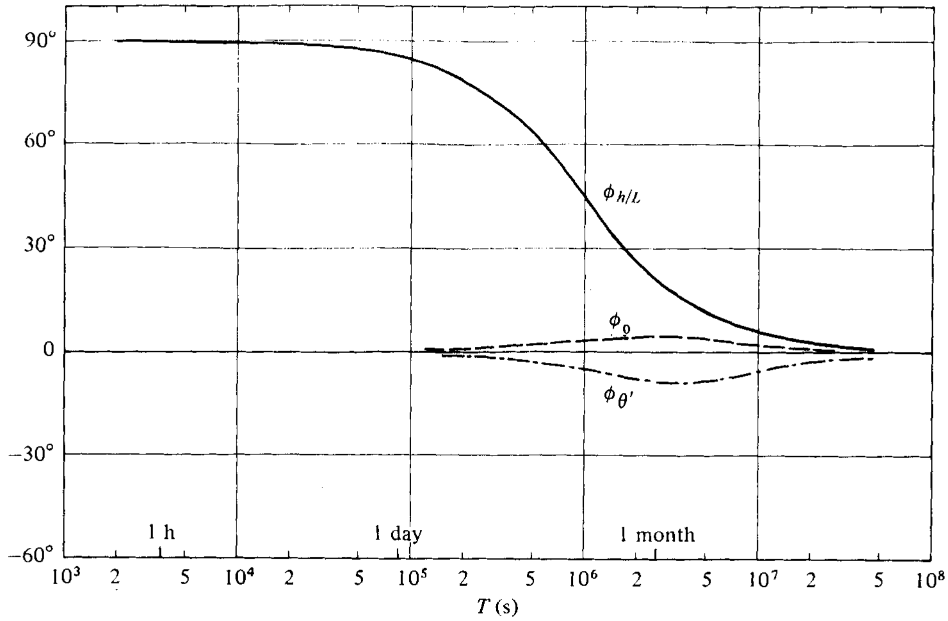

FIGURE 5. Variation in phases with T for $Q = 1\,\text{m}^3/\text{s}$, $I = 1{\cdot}5 \times 10^{-4}$ and $d = 0{\cdot}60\,\text{mm}$. Hence $h = 0{\cdot}47\,\text{m}$, $L = 8{\cdot}4\,\text{m}$ and $D = 2{\cdot}07\,\text{m}$.

where $\omega = 2\pi/T$ is the cyclic frequency and T the period. From (43) the step function $S(t)$ for $\beta^{-1}d(h/L)$ due to a sudden change in dQ is found to be

$$S(t) = 1 - \exp\left(-\frac{\alpha}{\beta}t\right),$$

from which the impulse response function $R(t)$ is obtained,

$$R(t) = \frac{d}{dt}[S(t)] = \frac{\alpha}{\beta}\exp\left(-\frac{\alpha}{\beta}t\right). \tag{49}$$

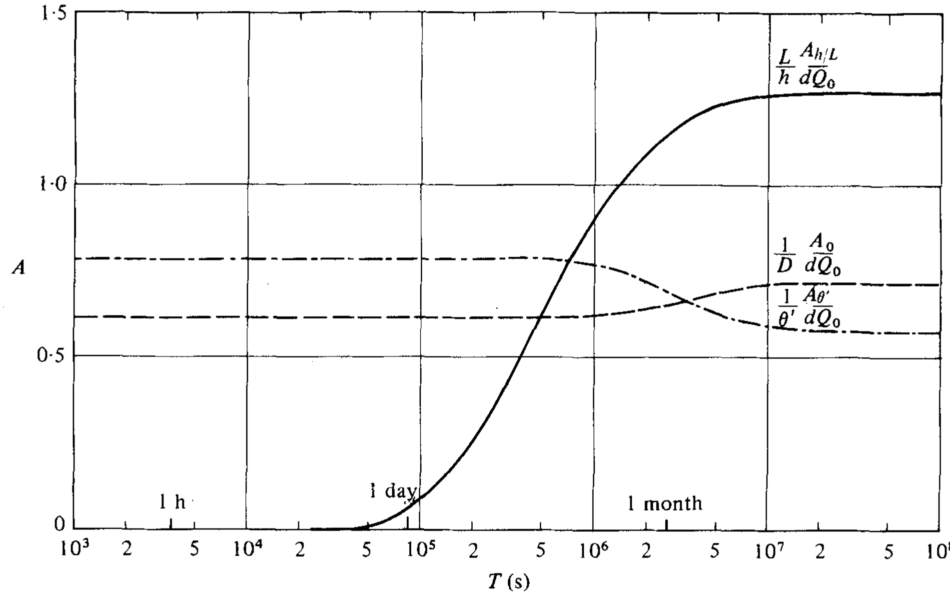

FIGURE 6. Variation in relative amplitudes with T for the data in figure 5.

Hence the variation in h/L is given by

$$\frac{1}{\beta}d\left(\frac{h}{L}\right) = \int_{-\infty}^{t} R(t-\tau)\,dQ_0 \sin(\omega\tau)\,d\tau,$$

or

$$d(h/L) = A_{h/L}\sin(\omega\tau - \phi_{h/L}), \tag{50}$$

where $$A_{h/L} = \frac{\beta dQ_0}{(1+\omega^2\beta^2/\alpha^2)^{\frac{1}{2}}} \quad \text{and} \quad \phi_{h/L} = \arctan\left(\frac{\omega\beta}{\alpha}\right).$$

From (45) the variation in water depth can be calculated in the same way:

$$dD = \frac{\partial D}{\partial Q}\,dQ_0 \sin(\omega t) + A_D^* \sin(\omega t - \phi_D^*), \tag{51}$$

where $$A_D^* = d_1\left[1+\left(\frac{\omega d_1}{\alpha L\,\partial D/\partial h}\right)^2\right]^{-\frac{1}{2}} \quad \text{and} \quad \phi_D^* = \arctan\left(\frac{\omega d_1}{\alpha L\,\partial D/\partial h}\right).$$

Equation (51) can easily be written as a new trigonometric function given by

$$dD = A_D \sin(\omega t - \phi_D). \tag{52}$$

Finally, the variation in the effective shear stress θ' can be calculated from (47) as

$$d\theta' = \rho_1 dQ_0 \sin(\omega t) + A_{\theta'}^* \sin(\omega t - \phi_{\theta'}^*), \tag{53}$$

where

$$A_{\theta'}^* = (\lambda-\rho_1)\left[1+\left(\frac{\omega(\lambda-\rho_1)^2}{\rho_2}\right)\right]^{-\frac{1}{2}} \quad \text{and} \quad \phi_{\theta'}^* = \arctan\left[\frac{\omega(\lambda-\rho_1)}{\rho_2}\right],$$

which again can be written as

$$d\theta' = A_{\theta'} \sin(\omega t - \phi_{\theta'}). \tag{54}$$

Two examples of phases and amplitudes obtained from (50), (52) and (54) are depicted as functions of the period T for two runs in figures 3–6. In figures 3 and 4 the data are for a smaller stream, e.g. a laboratory flume, while the data in figures 5 and 6 represent a larger stream.

In order to calculate the absolute value of the dune height h, water depth D and dune length L from these data the relation

$$fL/h = 0{\cdot}47 \tag{55}$$

is introduced. This relation is known to be valid for a dune-covered bed in equilibrium (Engelund & Hansen 1972). Equation (55) combined with (16) and (46) yields

$$5{\cdot}32\,\frac{h}{D}\exp\left(-2{\cdot}5\,\frac{h}{D}\right) = 1-0{\cdot}4\theta-\frac{0{\cdot}06}{\theta}. \tag{56}$$

From (56), (15), (16) and (46) the water depth and the dimensions of the dunes can be calculated if Q, I and d are known. The results are given in the figure captions.

The following conclusions can be drawn from the figures.

(i) The variations in h/L lag the variations in water discharge with a phase difference which varies from 90° for rapid changes to 0 for very slow changes in Q. $\phi_{h/L}$ is 45° for $T \sim 5\times10^3$ s or $1\frac{1}{2}$ h for the data given in figures 3 and 4. However, in larger streams, the time scale is increased, because of the larger dimensions of the bed forms. This is illustrated in figures 5 and 6, where $\phi_{h/L} = 45°$ for $T \sim 10^6$ s or about 12 days. For even larger streams, the time scale can be months or years. In these cases the present description must be regarded as only qualitative as other morphological processes may take place simultaneously. Finally, the time-scale is increased significantly if the Shields parameter approaches the critical value where the sediment transport vanishes. As $\phi_{h/L}$ varies between 0 and 90°, h/L is smaller for increasing than for decreasing $\phi_{h/L}$ for a fixed value of the water discharge. This is illustrated in figure 7(*a*), where the variation in Q and h/L with time is depicted for $T = 5\times10^3$ s, using the data from figures 3 and 4.

(ii) The phase difference between water depth and water discharge is zero at small and large values of T and has a maximum of about 10° between these two limits. Hence, like the ratio h/L, the water depth is smaller for increasing water discharge as depicted in figure 7(*b*). The reason for this is that the friction factor increases with the dune height, which lags the water discharge as described above.

(iii) The phase difference between the effective shear stress θ' (and hence the sediment transport) and the water discharge is negative between the two limits $T = 0$ and $T = \infty$, at which the phase is zero. Hence the sediment transport is largest for increasing water discharge; see figure 7(*c*). This is because the expansion loss behind the crest of the dune in this case is smaller owing to the smaller value of the dune height. It is typical for the runs described that the variation in the water discharge must take place so slowly that the dunes are able to travel a distance several times their own wavelength during the period T in order to react to the changes in the water discharge: in the run depicted in figure 3 the dune has travelled 6·3 times the wavelength for $T = 5\times10^3$ s, while the example given in figure 5 shows that the dune has travelled $3{\cdot}9L$ for $T = 10^6$ s.

The trends in the behaviour of the river given in figure 7 are confirmed by the measurements in the San Juan River depicted by Leopold, Wolman & Miller (1964,

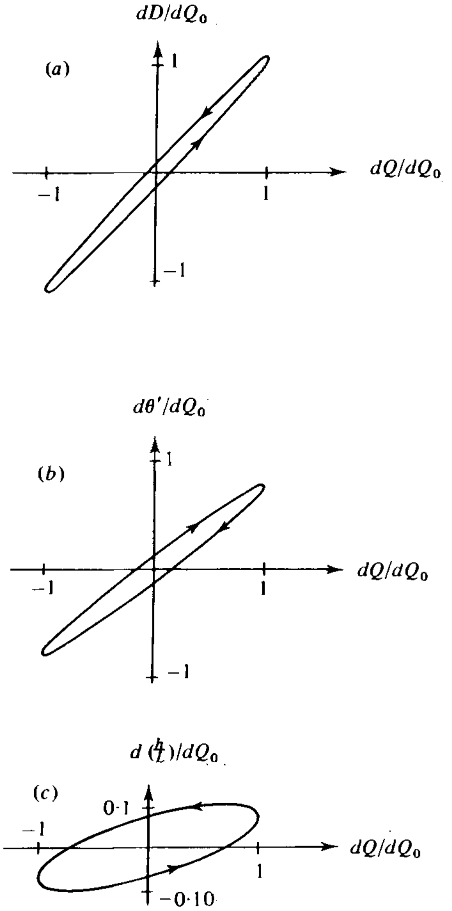

FIGURE 7. Variation in h/L, D and θ' with water discharge for $T = 5 \times 10^3$ s. The hydraulic data are the same as in figure 3.

figures 7–15). A direct comparison is not possible for two reasons: the grain size of the sediment is not reported and the variation in the water discharge is rather large. However, here also it is observed that for a given water discharge the water depth is less in the increasing state. Further, the transport of suspended sediment is larger in the increasing than in the decreasing state owing to the high value of θ' in the former case.

Observations reported by Carey & Keller (1957) concerning depth variations in the Mississippi River yield the same trend as that sketched in figure 7 (*b*), as do laboratory experiments carried out by Simons, Richardson & Haushild (1962). Here the variations in water discharge are also large, so a comparison with theory is not possible.

6. Conclusion

A theory has been developed which is able to predict the initial change in the dune height after a sudden change in water discharge. Bed load is assumed to be the dominant transport mechanism, and the square of the Froude number must be much smaller than unity.

By use of a step function, the behaviour of a river in which the water discharge varies weakly with time was studied. The theory predicts a smaller water depth for a given water discharge when the water discharge is increasing than when it is decreasing, and a larger sediment transport in the former state.

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